

Efficient Lipschitz Extensions for High-Dimensional Graph Statistics and Node Private Degree Distributions

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Abstract

Lipschitz extensions were recently proposed as a tool for designing node differentially private algorithms. However, efficiently computable Lipschitz extensions were known only for 1-dimensional functions (that is, functions that output a single real value). In this paper, we study efficiently computable Lipschitz extensions for multi-dimensional (that is, vector-valued) functions on graphs. We show that, unlike for 1-dimensional functions, Lipschitz extensions of higher-dimensional functions on graphs do not always exist, even with a non-unit stretch. We design Lipschitz extensions with small stretch for the sorted degree list and for the degree distribution of a graph. Crucially, our extensions are efficiently computable.

We also develop new tools for employing Lipschitz extensions in the design of differentially private algorithms. Specifically, we generalize the exponential mechanism, a widely used tool in data privacy. The exponential mechanism is given a collection of score functions that map datasets to real values. It attempts to return the name of the function with nearly minimum value on the data set. Our generalized exponential mechanism provides better accuracy when the sensitivity of an optimal score function is much smaller than the maximum sensitivity of score functions.

We use our Lipschitz extension and the generalized exponential mechanism to design a node-differentially private algorithm for releasing an approximation to the degree distribution of a graph. Our algorithm is much more accurate than algorithms from previous work.

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1 Introduction

The area of *differential privacy* studies how to output global information contained in a database while protecting privacy of individuals whose information it contains. Typically, the datasets considered are *tabular databases*, containing one row of information per person. While the area came a long way in the last decade in terms of the richness of information that can be released with differential privacy for tabular databases, we are lagging behind in our understanding of *graph* datasets that also contain relationships between various participants. Such datasets are used, for example, to capture relationships between people in a social network, communication patterns, and romantic relationships.

There are two natural variants of differential privacy that are suited for graph datasets: *edge* differential privacy and *node* differential privacy. Intuitively, the former protects relationships among individuals, while the latter protects each individual, together with all his/her relationships. Edge privacy is a weaker notion and has been studied more extensively, with algorithms now known for the release of subgraph counts and related scalar-valued functions [34, 35, 18, 32, 25, 19], the degree distribution [12, 13, 17, 24, 16], cut densities [11, 3] and the parameters of generative graph models [32, 19, 25, 16, 37]. Node differential privacy is a much stronger privacy guarantee, but is much harder to attain because it guards against larger changes in the input. Until recently, there were no known differentially private algorithms that gave accurate answers on sparse graphs, even for extremely simple statistics. In 2013, Blocki et al. [4], Kasiviswanathan et al. [20], Chen and Zhou [6] proposed two new techniques for node private algorithms: (i) using projections whose smooth sensitivity could be bounded (combined with mechanisms that add noise tailored to the smooth sensitivity [34]), and (ii) using *Lipschitz extensions* (combined with the standard Laplace mechanism). The latter technique yielded much more accurate algorithms than the former. In particular, it was used to obtain accurate node differentially private algorithms for computing subgraph counts and related statistics.

However, efficiently computable Lipschitz extensions were known only for 1-dimensional functions (that is, functions that output a single real value). In this paper, we study efficiently computable Lipschitz extensions for multi-dimensional (that is, vector-valued) functions. We show that, unlike for 1-dimensional functions, Lipschitz extensions of higher-dimensional functions do not always exist, even with a non-unit stretch. We design Lipschitz extensions with small stretch for the sorted degree list and for the degree distribution of a graph. Our extensions can be computed in polynomial time.

We also develop new tools for employing Lipschitz extensions in the design of differentially private algorithms. Specifically, we generalize the exponential mechanism of McSherry and Talwar [31], a widely used tool in data privacy. Our generalized mechanism provides better accuracy when the sensitivity of an optimal score function is much smaller than the maximum sensitivity of score functions.

We use our Lipschitz extension and the generalized exponential mechanism to design a node differentially private algorithm for releasing an approximation to the degree distribution of a graph. Our algorithm is much more accurate than those from previous work [4, 20].

Lipschitz extensions. Lipschitz extensions are basic mathematical objects studied in functional analysis.

Definition 1.1 (Lipschitz constant). Let $f : X \rightarrow Y$ be a function from a domain X to a range Y with associated distance measures d_X and d_Y . Function f has Lipschitz constant c (equivalently, is c -Lipschitz) if $d_Y(f(x), f(x')) \leq c \cdot d_X(x, x')$ for all $x, x' \in X$.

Definition 1.2 (Lipschitz extension). Consider a domain X and a range Y with associated distance measures d_X and d_Y , and let $X' \subset X$. Fix constants $c > 0$ and $s \geq 1$. Given a c -Lipschitz function $f' : X' \rightarrow Y$, a function $f : X \rightarrow Y$ is a Lipschitz extension of f' from X' to X with stretch s if

1. f is an extension of f' , that is, $f(x) = f'(x)$ on all $x \in X'$ and
2. f is $s \cdot c$ -Lipschitz.

If $s = 1$, then we call f a Lipschitz extension of f' from X' to X (omitting the stretch).

Functional analysts have devoted considerable attention to determining, for given metric spaces X, X' and Y , whether Lipschitz extensions with stretch 1 exist for all functions $f : X \rightarrow Y$. In contrast to this paper, the focus is mostly on continuous function spaces.

Lipschitz extensions of real-valued 1-dimensional functions with stretch 1 always exist [29]. We show that it is not true, in general, for multi-dimensional functions on graphs, even with non-unit stretch. The technical core of this paper is the construction of an efficiently computable extension of the degree distribution, a high-dimensional function on graphs, with small stretch.

Metrics on Graphs. Let \mathcal{G} denote the set of all finite labeled, unweighted undirected graphs. When the input data set is a graph in \mathcal{G} , there are two natural notions of “neighbor” (or adjacency). Two graphs G and G' are *edge neighbors* if they differ in one edge. Two graphs G and G' are *node neighbors* if one can be obtained from the other by removing one node and its adjacent edges. These two notions of neighbor induce two metrics on \mathcal{G} , node distance (d_{node}) and edge distance (d_{edge}).

Why are Lipschitz Extensions Useful for Privacy? A randomized algorithm \mathcal{A} is *node differentially private* if, for any two datasets that are “neighbors” in an appropriate sense, the *distributions* on the algorithms outputs are close in a multiplicative sense. Notions of stability and sensitivity play a key role in the design of differentially private algorithms. Differential privacy itself can be seen as a stability requirement, since the algorithm must map neighboring graphs to nearby distributions on outputs.

The two basic building blocks for designing differentially private algorithms, the Laplace and exponential mechanisms, rely on the *global sensitivity* of a function f , which is the Lipschitz constant of f viewed as a map from data sets (e.g., \mathcal{G} equipped with d_{node}) to ℓ_1^p (i.e., \mathbb{R}^p equipped with ℓ_1). The *Laplace mechanism* [10] shows that one can satisfy differential privacy by releasing $f(G)$ with additive noise proportional to the node global sensitivity in each coordinate.

The difficulty with employing the Laplace mechanism directly is that many useful functions on graphs are highly sensitive to the insertion or removal of a well-connected vertex. For example, the number of connected components of a graph may go from n to 1 with the insertion of a single vertex. The degree distribution of a graph can also change drastically, shifting up by 1 in every coordinate (as one vertex can increase the degree of all other vertices). This difficulty generally remains even if we shift from global sensitivity to more local notions (as in [34]) (roughly, interesting graphs such as those with low average degree are “near” other graphs with a vastly different value for the function).

One can get around this by focusing on a “nice” or “typical” subset of the space \mathcal{G} where the function f has low global sensitivity [4, 20, 6]. For example, let \mathcal{G}^D be the set D -bounded graphs, that is, graphs of maximum degree at most D . Many functions have bounded sensitivity (Lipschitz constant) on \mathcal{G}^D . The number of triangles in a graph, for instance, changes by at most $\binom{D}{2}$ among node-neighboring graphs of degree at most D , and the degree list changes by at most $2D$ in ℓ_1 .

Given a function f that has low Lipschitz constant on “nice” graphs, if we find an efficiently computable Lipschitz extension \hat{f} that is defined on all of \mathcal{G} , then we can use the Laplace mechanism to release $\hat{f}(G)$ with relatively small additive noise. The lower the stretch of the extension, the lower the overall noise. The result will be accurate when the input indeed falls into, or near, the class of “nice” graphs. Interestingly, the class of “nice” graphs need not contain the input for the answer to be accurate—in our main application, we use \mathcal{G}^D as the set of “nice” graphs, but D is set much lower than the actual maximum degree of the input.

Existence and efficiency of Lipschitz extensions. Motivated by this methodology, we ask: when do Lipschitz extensions exist, and when do they admit efficient algorithms? The existence question has drawn interest from functional analysis and combinatorics for nearly a century [29, 21, 36, 27, 15, 28, 1, 2, 22, 33, 23, 26]; see Lee and Naor [23] for an overview. For any real-valued function $f : \mathcal{G}^D \rightarrow \mathbb{R}$, there exists an extension $\hat{f} : \mathcal{G} \rightarrow \mathbb{R}$ whose node sensitivity is the same as that of f . Kasiviswanathan et al. [20], Chen and Zhou [6] constructed *polynomial-time* computable Lipschitz extensions from \mathcal{G}^D to \mathcal{G} of several real-valued functions on graphs. The techniques in [4, 20, 6] apply to functions that count structures in a graph, possibly with weights (for example, the number of edges in a graph, the number of triangles in a simple graph; in a graph where vertices and edges have attributes, one could count edges that link nodes labeled by different genders in a social network, or triangles involving vertices labeled with different scientific fields in a collaboration graph).

Prior work on constructions of higher-dimensional extensions focused on extending functions on a metric space X , where X is given explicitly as input (say, as a distance matrix) [23, 26]. Such constructions can, at best, run in time polynomial in the size of X . The size of \mathcal{G}^D is infinite, and even restricting to graphs on at most n vertices leaves a set that is exponentially large in n . Moreover, generic constructions have stretch at least polynomial in the log of the metric’s cardinality, at least \sqrt{n} in our case.

1.1 Our Contributions

In this paper, we demonstrate that efficient and nontrivial constructions of Lipschitz extensions for high-dimensional graph summaries are possible. We also develop new machinery for using these extensions in the context of differentially private algorithms.

Lipschitz Extension of the Degree List (Section 4). Our main technical contribution is a polynomial-time, constant-stretch Lipschitz extension of the *sorted degree list*, viewed as a function from \mathcal{G}^D to ℓ_1^* , to all of \mathcal{G} . Here ℓ_1^* denotes the ℓ_1 metric on the space of finite-length real sequences, where sequences of different length are padded with 0’s to compute distance.

Given an arbitrary graph G , our function $\hat{f}_D(G)$ outputs a nonincreasing real sequence of length $|V_G|$. If the maximum degree of G is D or less, the output is the sorted list of degrees in G . The output can be thought of as a list of “fractional degrees”, where “fractional edges” are real weights

in $[0, 1]$ and the “fractional degree” of a vertex is the sum of the weights of its adjacent edges. The weights are selected by minimizing a quadratic function over the polytope of s - t flows in a directed graph closely related to G . Previous work [20] had shown that the *value* of the maximum flow in the graph has low sensitivity; by introducing the quadratic penalty, we give a way to select an optimal flow that changes slowly as the graph itself changes. Introducing a strongly convex penalty (or *regularizer*) to make the solution of an optimization problem stable to changes in the loss function is common in machine learning. In our setting, however, it is the *constraints* of the convex program that change with data, and not the loss function.

Theorem 1.3. *There is a Lipschitz extension of deg-list, viewed as a function taking values in ℓ_1^* , from \mathcal{G}^D to \mathcal{G} with stretch $3/2$ that can be computed in polynomial time.*

The sorted degree list has ℓ_1 sensitivity D on \mathcal{G}^D . The extension $\hat{f}_D(G)$ has ℓ_1 sensitivity at most $3D$ (the stretch is thus at most $3/2$). Previous results on Lipschitz extensions only imply the existence of an extension with stretch at least n ; see Section 3 for discussion of the general results.

We use our Lipschitz extension of the sorted degree list to get a Lipschitz extension of the degree distribution (a list of counts of nodes of each degree) and the degree CDF (a list of counts of nodes of at least each given degree). These functions condense the information to a D -dimensional vector (regardless of the size of the graph), making it easier to release with node-differential privacy.

Generalized Exponential Mechanism for Scores of Varying Sensitivity (Section 5).

One of the difficulties that arises in using Lipschitz extensions for differentially private algorithms is selecting a good class of inputs from which to extend. For example, to apply our degree distribution extension, we must select the degree bound D . More generally, we are given a collection of possible extensions $\hat{f}_1, \dots, \hat{f}_k$, each of which agrees with f on a different set and has different sensitivity Δ_i .

For a large class of extensions, we can abstract the task we are faced with as a private optimization problem: given a set of *real-valued* functions q_1, \dots, q_k , the goal is to output the index \hat{i} of a function with approximately minimal value on the data set x (so that $q_{\hat{i}}(x) \approx \min_i q_i(x)$). (In our setting, the q_i functions are related to the error of the approximation \hat{f}_i on x). Suppose that each q_i has a known (upper bound on) global sensitivity Δ_i . The *error* of an output \hat{i} on input x is the difference $q_{\hat{i}}(x) - \min_i q_i(x)$.

The exponential mechanism of McSherry and Talwar [31], a widely used tool in differentially private algorithms, achieves error that scales with the *largest* of the sensitivities. Specifically, for every $\beta > 0$, with probability $1 - \beta$, the output \hat{i} satisfies $q_{\hat{i}}(x) \leq \min_i q_i(x) + \Delta_{max} \cdot \frac{2 \log(k/\beta)}{\epsilon}$ where $\Delta_{max} = \max_i \Delta_i$.

In contrast, we give an algorithm whose accuracy scales with the sensitivity of the *optimal* score function Δ_{i^*} where $i^* = \operatorname{argmin}_i q_i(x)$. Our mechanism requires as input an upper bound $\beta > 0$ on the desired probability of a “bad” outcome; the algorithm’s error guarantee depends on this β .

Theorem 1.4 (Informal). *For all settings of the input parameters $\beta \in (0, 1)$, $\epsilon > 0$, the Generalized Exponential Mechanism is ϵ -differentially private. For all inputs x , the output \hat{i} satisfies*

$$q_{\hat{i}}(x) \leq \min_i \left(q_i(x) + \Delta_i \cdot \frac{4 \log(k/\beta)}{\epsilon} \right).$$

This guarantee can be much tighter than that of the usual exponential mechanism. For instance, in our setting, the Δ_i ’s grow exponentially with i yet on sparse graphs, the best choice of Δ_i is

for i relatively small. (Also, the issue is not merely with the error guarantee. The exponential mechanism provides bad outputs for many inputs where the true minimizer has low sensitivity.)

We can use our algorithm for selecting the sensitivity parameter for the Lipschitz extensions of graph functions in [4, 20, 6] and in this work. (These parameters are sometimes interpretable as a degree bound, as in the case of the degree distribution, but not always; for example, when computing the number of triangles, the parameter is a bound on the number of triangles involving any one vertex). This allows the algorithm to adapt to the specific input. The guarantee we get is that the error of the overall algorithm (that is approximating some function of an n -node graph) is at most $O(\log \log n)$ times higher than one would get with the best Lipschitz constant. In contrast, the parameter selection method of Chen and Zhou [6] provides only a $O(\log n)$ guarantee on the error blow-up, and is specific to the extensions they construct.

Differentially Private Algorithms for Releasing the Degree Distribution (Section 6).

We can combine the Lipschitz extension of the degree list and the parameter selection algorithm to get a differentially private mechanism for releasing the degree distribution of a graph that automatically adapts to the structure of the graph.

We show that our algorithm provides an accurate estimate on a large class of graphs, including graphs with low average degree whose degree distribution is heavy-tailed. We measure accuracy in the ℓ_1 norm, normalized by the number of nodes in the graph — i.e., we deem the algorithm accurate if the total variation distance between the true degree distribution and the estimate is small.

This measure goes to 0 for graphs of low average degree in which the tail of the degree distribution decreases slightly more quickly than what trivially holds for all graphs. If \bar{d} is the average degree in a graph, Markov’s inequality implies that the fraction of nodes with degree above $t \cdot \bar{d}$ is at most $1/t$. We assume that this fraction goes down as $1/t^\alpha$ for a constant $\alpha > 1$. The condition is called α -decay. Our algorithm need not be given α or the average degree of the graph; these are implicitly taken into account by parameter selection. Our assumption is satisfied by all the well-studied social network models we know of, including so-called *scale-free* graphs [7].

2 Definitions

Notation. We use $[n]$ to denote the set $\{1, \dots, n\}$. For a graph, (V, E) , $\bar{d}(G) = 2|E|/|V|$ is the average degree of the graph G and $\deg_v(G)$ denotes the degree of node $v \in V$ in G . When the graph referenced is clear, we drop G in the notation. The asymptotic notation $O_n(\cdot)$, $o_n(\cdot)$ is defined with respect to growing n . Other parameters are assumed to be functions independent of n unless specified otherwise.

2.1 Graphs Metrics and Differential Privacy

Definition 2.1 ((ϵ, δ) -edge/node-privacy). *A randomized algorithm \mathcal{A} is (ϵ, δ) -edge-private (respectively, node-private) if for all events S in the output space of \mathcal{A} , and edge (respectively, node) neighbors G_1, G_2 ,*

$$\Pr[\mathcal{A}(G_1) \in S] \leq \exp(\epsilon) \times \Pr[\mathcal{A}(G_2) \in S] + \delta.$$

When $\delta = 0$, the algorithm is ϵ -edge-private (respectively, ϵ -node-private). In this paper, if node or edge privacy is not specified, we mean node privacy by default.

For simplicity of presentation, we assume that $n = |V|$, the number of nodes of the input graph G , is publicly known. This assumption is justified since, as we will see, one can get an accurate estimate of $|V|$ by running a node-private algorithm.

Both variants of differential privacy “compose” well, in the sense that privacy is preserved (albeit with slowly degrading parameters) even when the adversary gets to see the outcome of multiple differentially private algorithms run on the same data set.

Lemma 2.2 (Composition, post-processing [30, 8]). *If an algorithm \mathcal{A} runs t randomized algorithms $\mathcal{A}_1, \dots, \mathcal{A}_t$, each of which is (ϵ, δ) -differentially private, and applies a randomized algorithm g to the outputs, i.e., $\mathcal{A}(G) = g(\mathcal{A}_1(G), \dots, \mathcal{A}_t(G))$, then \mathcal{A} is $(t\epsilon, t\delta)$ -differentially private.*

2.2 Basic Tools

Global Sensitivity and the Laplace Mechanism. In the most basic framework for achieving differential privacy, Laplace noise is scaled according to the *global sensitivity* of the desired statistic f . This technique extends directly to graphs as long as we measure sensitivity with respect to the metric used in the definition of the corresponding variant of differential privacy. Below, we explain this (standard) framework in terms of node privacy. Let \mathcal{G} denote the set of all graphs.

Definition 2.3 (Global Sensitivity [10]). *The ℓ_1 -global node sensitivity of a function $f : \mathcal{G} \rightarrow \mathbb{R}^p$ is:*

$$\Delta f = \max_{G_1, G_2 \text{ node neighbors}} \|f(G_1) - f(G_2)\|_1.$$

Equivalently, Δf is the Lipschitz constant of a function viewed as a map from $(\mathcal{G}, d_{\text{node}})$ to ℓ_1^p .

For example, the number of edges in an n -node graph has node sensitivity n , since adding or deleting a node and its adjacent edges can add or remove at most n edges. In contrast, the number of nodes in a graph has node sensitivity 1.

A Laplace random variable with mean 0 and standard deviation $\sqrt{2}\lambda$ has density $h(z) = (1/(2\lambda))e^{-|z|/\lambda}$. We denote it by $\text{Lap}(\lambda)$.

Theorem 2.4 (Laplace Mechanism [10]). *The algorithm $\mathcal{A}(G) = f(G) + \text{Lap}(\Delta f/\epsilon)^p$ (which adds i.i.d. noise $\text{Lap}(\Delta f/\epsilon)$ to each entry of $f(G)$) is ϵ -node-private.*

Thus, we can release the number of nodes $|V|$ in a graph with noise of expected magnitude $1/\epsilon$ while satisfying node differential privacy. Given a public bound n on the number of nodes, we can release the number of edges $|E|$ with additive noise of expected magnitude n/ϵ .

Exponential Mechanism. Suppose data sets are members of a universe U equipped with a neighbor relation (for example, $U = \mathcal{G}$ with vertex neighbors). Suppose we are given a collection of functions q_1, \dots, q_k , from U to \mathbb{R} such that for each $i \in [k]$, the function $q_i(\cdot)$ has sensitivity at most Δ . The exponential mechanism (McSherry and Talwar [31]) takes a data set and aims to output an index \hat{i} for which $q_{\hat{i}}(G)$ has nearly minimal value at G , that is, such that $q_{\hat{i}}(G) \approx \min_i q_i(G)$. The algorithm \mathcal{A} samples an index i such that $\Pr(\mathcal{A}(G) = i) \propto \exp\left(\frac{\epsilon}{2\Delta} q_i(G)\right)$.

Lemma 2.5 (Exponential Mechanism [31]). *The algorithm \mathcal{A} is ϵ -differentially private. Moreover, with probability at least $1 - \eta$, its output \hat{i} satisfies $q_{\hat{i}}(G) \leq \min_i (q_i(G)) + \frac{2\Delta \ln(k/\eta)}{\epsilon}$.*

There is a simple, efficient implementation of the exponential mechanism that adds exponential noise to each score function and reports the maximizer of the noisy scores (see, e.g., [9, Sec. 3.4]).

3 General Results on Lipschitz Extensions

A number of basic results from functional analysis apply to our setting. Let ℓ_p^d denote the set \mathbb{R}^d equipped with the ℓ_p metric.

When $Y = \mathbb{R}$ (with the usual metric), a Lipschitz extension always exists [29]. The classic construction, given a c -Lipschitz function $f : X \rightarrow \mathbb{R}$, defines $\hat{f} : X' \rightarrow \mathbb{R}$ as

$$\hat{f}(y) = \inf_{x \in X} (f(x) + c \cdot d_{X'}(x, y)) .$$

The function \hat{f} is also c -Lipschitz, but need not necessarily be easy to compute even if f admits efficient algorithms.

Blocki et al. [4], Kasiviswanathan et al. [20], Chen and Zhou [6] constructed *polynomial-time* Lipschitz extensions from \mathcal{G}^D to \mathcal{G} of several real-valued functions on graphs (see Introduction).

In this work, our focus is on higher-dimensional functions on graphs, i.e., functions that map graphs into \mathbb{R}^p for $p > 1$. As with one-dimensional functions, there always exist stretch-1 extensions of functions that take values in ℓ_∞^p for any dimension p , since one can separately find an extension for each coordinate of f . It is also true for ℓ_1^2 , since ℓ_1^2 is isomorphic to ℓ_∞^2 . However, stretch-1 extensions need not exist when $Y = \ell_2^p$ or ℓ_1^p for larger p . There is a growing body of theory on the minimal stretch required for extensions among different spaces; see [2, 23] for a concise summary of known general results on the problem.

Our first result is that one cannot always get stretch-1 extensions for functions from \mathcal{G}^D to ℓ_1^p or ℓ_2^p . We prove it at the end of this section. It is the only lower bound on extendability for these metrics we are aware of.

Proposition 3.1. *Consider the vertex distance on \mathcal{G} . There is an absolute constant $c > 1$ such that: (1) for all $p \geq 3$, there exist symmetric functions from \mathcal{G}^D to ℓ_1^p that do not admit a stretch- c extension to \mathcal{G} ; (2) for all $p \geq 2$, there exist symmetric functions from \mathcal{G}^D to ℓ_2^p that do not admit a stretch- c extension to \mathcal{G} .*

This lower bound extends to edge distance on \mathcal{G} (we omit the proof). Moreover, for edge distance, it is essentially tight: a result of Blocki et al. [4] on smooth projections implies that every function on \mathcal{G}^D which is Lipschitz under the *edge distance* metric on \mathcal{G}^D can be extended to all of \mathcal{G} with stretch at most 3, regardless of the output metric. However, the construction does not apply to vertex distance on graphs.

For the *vertex distance* on \mathcal{G}^D , known results yield extensions with stretch that is polynomial in either p or n (the size of the graph). We outline these briefly: Lee and Naor [23, Theorem 1.6] show that one can get extensions with stretch $O(\rho(X))$, where $\rho(X)$ is the *doubling dimension* of the metric space X (in our case, \mathcal{G}^D or \mathcal{G}_n^D). Unfortunately, the vertex metric on \mathcal{G}_n^D has doubling dimension at least n , even for $D = 4$ and even if we identify isomorphic graphs (see Appendix A for formal definitions and a proof). Makarychev and Makarychev [26] show that functions from any metric on N points can be extended to an arbitrary containing space with stretch at most $O(\log N / \log \log N)$. Since $\log N$ is approximately nD for \mathcal{G}_n^D , this again yields large stretch. Finally, another general approach, based on the dimension of the image space, yields stretch p and \sqrt{p} for maps into ℓ_1^p and ℓ_2^p respectively (in our case, one can obtain this by separately extending each of the p coordinates of the output).

Proof of Proposition 3.1. Our proof is inspired by the example of Benyamini and Lindenstrauss [2] of spaces $X \subseteq X'$ and a function $f : X \rightarrow \ell_2^2$ such that there is no stretch-1 extension of f from X to X' .

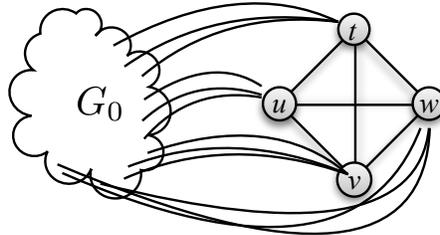
We start with the case of maps into ℓ_1 . Let X' denote the metric space $\{a, b, c, d, e\}$ with all pairwise distances among $X = \{a, b, c, d\}$ equal to 2, and distances $d_{X'}(x, e) = 1$ for $x \in X$ (pictured as a graph below). Consider the function $f : X \rightarrow \mathbb{R}^3$ that maps X to the corners of a particular tetrahedron:

$$\begin{aligned}
 f(a) &= \begin{pmatrix} -1, & 1, & 1 \end{pmatrix} \\
 f(b) &= \begin{pmatrix} 1, & -1, & 1 \end{pmatrix} \\
 f(c) &= \begin{pmatrix} 1, & 1, & -1 \end{pmatrix} \\
 f(d) &= \begin{pmatrix} -1, & -1, & -1 \end{pmatrix}
 \end{aligned}
 \tag{1}$$

The function f is 2-Lipschitz if we view the image as ℓ_1^3 , but there is no way to extend it to all of X' in either metric without stretching the Lipschitz constant. To satisfy the Lipschitz constraint $f(e)$ has to be exactly halfway between every pair in the set $\{f(a), f(b), f(c), f(d)\}$ (since it has to be at distance at most 2 from each of the points). The points that are halfway from a to b have third coordinate 1; the points halfway from c to d have third coordinate -1; there is no intersection between the two sets, and hence no possible value for $f(e)$. Any value for $f(e)$ results in a stretch of at least some absolute constant $c > 1$.

We can lift this example to other domains $X \subset X'$. For example, we can take X' to be ℓ_1^4 , and let $a = (1, 0, 0, 0), b = (0, 1, 0, 0), c = (0, 0, 1, 0), d = ((0, 0, 0, 1)$ and $e = (0, 0, 0, 0)$.

Lifting the example to $\mathcal{G}^D \subset \mathcal{G}$ is a bit messier. Fix d at least 4. Let G_0 be a graph on at least $4(d-2)$ vertices with maximum degree at most $d-1$ and no nontrivial automorphisms (a sufficiently large random graph satisfies the criteria with high probability [5, Chap. 9]). We create a larger graph H by adding four vertices $\{t, u, v, w\}$ to G_0 , among which all possible edges exist, and such that t, u, v and w are connected to disjoint subsets of $d-2$ vertices in G_0 (this is possible since G_0 must have at least $4(d-2)$ vertices). The vertices t, u, v, w have degree $d+1$ in H .



To embed our counterexample in \mathcal{G} , let $e = H$, and let $\{a, b, c, d\}$ be the graphs obtained by deleting one of t, u, v, w (respectively) from H . The four graphs a, b, c, d lie in \mathcal{G}^D , and no pair of them is isomorphic (since u, v, w are connected to disjoint sets of a graph with no automorphisms). The vertex distance between any pair of graphs in a, b, c, d is 2, and their distance from e is 1. We can set the values of f on a, b, c, d as in (1) (this is consistent with the requirement that f be symmetric since the graphs are not isomorphic). By the reasoning above, f is 2-Lipschitz but there is no way to assign a value to $f(e)$ without increasing the stretch of f .

We must still show that it is possible to assign values to functions on the remaining graphs in \mathcal{G}^D without increasing the Lipschitz constant.

For the graphs G that can be obtained by exactly two of t, u, v, w (along with corresponding edges) to G_0 , there two of the graphs in $\{a, b, c, d\}$ that are at distance 1 from G . We set $f(G)$ to be the average of the values of f at these two nearest graphs (for example, $G + \{t, u\}$ is at distance 1 from graphs c and d ; we set $f(G + \{t, u\}) = (0, 0, -1)$). Note that $f(G + \{t, u\})$ is at distance 2 from $f(c)$ and $f(d)$, as required. For all other graphs, $G \in \mathcal{G}^D$, we set $f(G) = (0, 0, 0)$. One can verify by inspection that the 2-Lipschitz property is satisfied on all of \mathcal{G}^D by f .

Finally, we note that an even simpler example works for maps into ℓ_2^2 . Starting with the same spaces X and X' , we can consider a function $f : X \rightarrow \mathbb{R}^2$ that maps $\{a, b, c\}$ to the corners of an equilateral triangle with side-length 1. The map is $\frac{1}{2}$ -Lipschitz on $\{a, b, c\}$, but cannot be extended to all of X' (since there is no point at distance $\frac{1}{2}$ of all three corners. Lifting the example to $\mathcal{G}^D \subset \mathcal{G}$ is similar to the ℓ_1 case. \square

4 Lipschitz Extensions of the Degree List and Distribution

4.1 Lipschitz Extension of the Degree List

In this section, we give a Lipschitz extension of the degree list. For an n -node graph G , let

$$\text{deg-list}(G) = \text{sort}(\text{deg}_1(G), \dots, \text{deg}_{|V(G)|}(G))$$

denote the list of degrees of G sorted in nonincreasing order.

We view the degree list as an element of \mathbb{R}^* (the set of finite sequences of real numbers). We equip the space with the ℓ_1 distance, where the sequences of different lengths are padded with 0's to allow comparison. This representation is convenient for handling node additions and deletions.

The global ℓ_1 sensitivity (under node insertion and removal) of the degree list on D -bounded graphs is $2D$ because the unsorted degree list has sensitivity $2D$ and, as Hay et al. [12] observed, sorting does not increase the ℓ_1 distance between vectors. We construct an extension that agrees with deg-list on \mathcal{G}^D and has global sensitivity at most $3D$.

Before explaining our construction, we consider a simpler “straw man” construction to illustrate the problem’s difficulty: suppose that given the degree list $\text{deg-list}(G)$, we obtain $\hat{f}_D(G)$ by rounding all degrees above D down to D . This will not affect the degrees in graph with maximum degree D , but it is not $O(D)$ Lipschitz: consider a star graph on n vertices, with one vertex of degree $n - 1$ and $n - 1$ vertices of degree 1. Simple rounding would report $\hat{f}(G)$ as $(D, 1, \dots, 1)$. But the graph has a neighbor G' with no edges at all, for which the reported degree list would be all 0's. Those vectors differ by $n + D - 1$ in the ℓ_1 norm. One can try simple ways of dropping very high-degree vertices (an idea called “projection” in [4, 20]), but those do not yield uniform bounds on the sensitivity of the resulting degree sequence and result in more noise being added for privacy.

Like in [20], our starting point is the construction of the flow graph G' for graph G . Kaviswanathan et al. [20] proved that the value of the maximum flow in G' is a Lipschitz extension of the number of edges in G . We will use the flow values on certain edges as a proxy for degrees of related vertices. The main challenge is that, whereas the value of the maximum flow in G' is unique, the actual flow on specific edges is not.

Definition 4.1 (Flow graph). *Given a graph $G = (V, E)$, let $V_\ell = \{v_\ell \mid v \in V\}$ and $V_r = \{v_r \mid v \in V\}$ be two copies of V , called the left and the right copies, respectively. Let D be a natural number less than n . The flow graph of G with threshold D , a source s and a sink t is a directed graph on nodes $V_\ell \cup V_r \cup \{s, t\}$ with the following capacitated edges: edges of capacity D from the source s to*

all nodes in V_ℓ and from all nodes in V_r to the sink t , and unit-capacity edges (u_ℓ, v_r) for all edges (u, v) of G . The flow graph of G is denoted $\text{FG}(G)$.

We would like our extension function to output the sorted list of flows leaving the source vertex in some maximum flow. The challenge is that there may be many maximum flows. If we select a maximum flow arbitrarily, then the selected flow may be very sensitive to changes in the graph, even though its value changes little. We get around this by selecting a flow that minimizes a strictly convex function of the flow values.

Definition 4.2 (Lipschitz extension of degree list). *Given a flow f of $\text{FG}(G)$, let $f(e)$ denote the flow on an edge e . Also, let $f_{s\bullet}$ be the vector of flows on the edges leaving the source s , let $f_{\bullet t}$ be the vector of flows on the edges entering t , and let $f_{s\bullet, \bullet t}$ be the concatenation of the two vectors. We use \vec{D}_{2n} to denote a vector of length $2n$, where all entries are D . Let $\Phi(f)$ be the squared ℓ_2 distance between $f_{s\bullet, \bullet t}$ and \vec{D}_{2n} , that is,*

$$\Phi(f) = \|f_{s\bullet, \bullet t} - \vec{D}_{2n}\|_2^2 = \sum_{v \in V} ((D - f(s, v_\ell))^2 + (D - f(v_r, t))^2).$$

Let f be the flow that minimizes the objective function Φ over all feasible flows in $\text{FG}(G)$. Define $\hat{f}_D(G)$ to be the sorted list of flows along the edges leaving the source, that is, $\hat{f}_D(G) = \text{sort}(f_{s\bullet})$.

The function $\hat{f}_D(G)$ is uniquely defined because the objective Φ is strictly convex in the values $f_{s\bullet, \bullet t}$. $\hat{f}_D(G)$ can be approximated to arbitrary precision in polynomial time, since it is the minimum of a strongly convex function over a polytope with polynomially many constraints. The approximation may slightly increase the sensitivity; in our application, one can account for this by adding slightly more than $3D/\epsilon$ noise in each coordinate.

Theorem 1.3 follows from the following theorem.

Theorem 4.3. *The function $\hat{f}_D(G)$ is a Lipschitz extension of $\text{deg-list}(G)$ from \mathcal{G}^D to \mathcal{G} of stretch $3/2$. In other words,*

1. *If G is D -bounded, then $\hat{f}_D(G) = \text{deg-list}(G)$.*
2. *For any two graphs G_1, G_2 (not necessarily D -bounded) that are node neighbors,*

$$\|\hat{f}_D(G_1) - \hat{f}_D(G_2)\|_1 \leq 3D.$$

Proof of Theorem 4.3 (item 1). The flow that assigns 1 to all edges (u_ℓ, v_r) and $\text{deg}(v)$ to all edges (s, v_ℓ) and (v_r, t) strictly dominates all feasible flows. In particular, it minimizes Φ since, for $x \in [0, D]$, function $(D - x)^2$ is decreasing in x . \square

There are two distinct notions of optimality of a flow in $\text{FG}(G)$: optimality with respect to Φ , which we call Φ -optimality, and optimality of the net flow from s to t , called *net flow optimality*. Next, we show that Φ -optimality implies net flow optimality.

Lemma 4.4. *For every graph G , if f minimizes Φ among valid flows for the flow graph $\text{FG}(G)$, then f has maximum net flow from s to t in $\text{FG}(G)$.*

Proof. If f does not have maximum net flow, then we can find a shortest augmenting path p from s to t . Let $c > 0$ be the minimal residual capacity of the edges in p . Since p is a shortest path, it is simple; thus, adding cp to f results in a feasible flow, but does not decrease the flow along any edge leaving s or entering t . This implies that $\Phi(f + cp) < \Phi(f)$ (since Φ is strictly decreasing in each argument), contradicting the Φ -optimality of f . \square

The flow graph $\text{FG}(G)$ admits a simple symmetry: for any flow f , we can obtain a feasible flow $\pi(f)$ by swapping the roles of s and t and the roles of left and right copies of all vertices. That is, we define $\pi(f)(s, v_\ell) := f(v_r, t)$, $\pi(f)(u_r, t) := f(s, u_\ell)$, $\pi(f)(u_\ell, v_r) := f(v_\ell, u_r)$ for all vertices v, u in G . Flow f is *symmetric* if $\pi(f) = f$. For every graph G , there exists a symmetric Φ -optimal flow in $\text{FG}(G)$: given any Φ -optimal flow f' , the flow $f'' = \frac{1}{2}(f' + \pi(f'))$ is symmetric, feasible (because the set of feasible flows is convex) and has objective value at most $\Phi(f')$ by convexity of Φ .

Proof of Theorem 4.3 (item 2). Suppose a graph G_1 on n vertices is obtained by removing a node v^{new} along with its associated edges from a graph G_2 (on $n + 1$ vertices).

Let f_1, f_2 be Φ -optimal symmetric flows for the flow graphs $\text{FG}(G_1)$ and $\text{FG}(G_2)$, respectively.

Observe that f_1 is a feasible flow in $\text{FG}(G_2)$. Consider the flow $\Delta = f_2 - f_1$. Note that Δ is a maximum signed flow in the residual graph of flow f_1 for $\text{FG}(G_2)$. In particular, Δ satisfies flow and capacity constraints, but not necessarily positivity. Since $\|\hat{f}_D(G_1) - \hat{f}_D(G_2)\|_1 = \|\Delta_{s\bullet}\|_1$, our goal is to prove $\|\Delta_{s\bullet}\|_1 \leq 3D$.

Next, we decompose Δ into three *subflows*. A *subflow* of a flow Δ is a flow Δ' such that for all edges e , the flows $\Delta(e)$ and $\Delta'(e)$ cannot have different signs and $|\Delta'(e)| \leq \Delta(e)$. We start by decomposing Δ into subflows that form simple s - t paths and simple cycles. Then we group them as follows:

- Let Δ^s be the sum of all flows from the initial decomposition that form paths and cycles using the edge (s, v_ℓ^{new}) .
- Let Δ^t be the sum of all flows from the initial decomposition that form paths and cycles using the edge (v_r^{new}, t) , but not (s, v_ℓ^{new}) .
- Let Δ^0 be the sum of the remaining flows, i.e., $\Delta^0 = \Delta - \Delta^s - \Delta^t$.

Since, by definition of the subflow decomposition, $\|\Delta_{s\bullet}\|_1 = \|\Delta^s_{s\bullet}\|_1 + \|\Delta^t_{s\bullet}\|_1 + \|\Delta^0_{s\bullet}\|_1$, it remains to bound the three values in the sum. We do it in the following three lemmas.

Lemma 4.5. $\|\Delta^s_{s\bullet}\|_1 \leq 2D$.

Proof. Recall that Δ^s can be decomposed into simple s - t paths and simple cycles that use the edge (s, v_ℓ^{new}) . Each such path contributes the value of its flow to $\|\Delta^s_{s\bullet}\|_1$, and each such cycle contributes at most twice the value of its flow. Since the total flow $\Delta^s(s, v_\ell^{\text{new}})$ is at most D , we get that $\|\Delta^s_{s\bullet}\|_1 \leq 2D$. \square

Lemma 4.6. $\|\Delta^t_{s\bullet}\|_1 \leq D$.

Proof. Recall that Δ^t can be decomposed into simple s - t paths and cycles that use the edge (v_r^{new}, t) , but not (s, v_ℓ^{new}) . Each such path contributes the value of its flow to $\|\Delta^t_{s\bullet}\|_1$. Any such cycle contributes 0 to $\|\Delta^t_{s\bullet}\|_1$ because any simple cycle in Δ that starts from t cannot reach s . If it did,

one could find an augmenting s - t path in Δ , implying that f_2 is not a net value optimal flow in $\text{FG}(G_2)$ and, by Lemma 4.4, contradicting Φ -optimality of f_2 in $\text{FG}(G_2)$.

Since the total flow $\Delta^t(v_r^{\text{new}}, t)$ is at most D , we get that $\|\Delta^t_{s\bullet}\|_1 \leq D$. \square

Lemma 4.7. $\|\Delta^0_{s\bullet}\|_1 = 0$.

Proof. The flow Δ^0 does not use the edges (s, v_ℓ^{new}) and (v_r^{new}, t) since all flow in Δ along (s, v_ℓ^{new}) and (v_r^{new}, t) has been used by $\Delta^s + \Delta^t$. Consequently, Δ^0 has no flow passing through v_ℓ^{new} and v_r^{new} . Therefore, Δ^0 is a feasible flow for the residual graph of f_1 in $\text{FG}(G_1)$. We conclude that $f_1 + \Delta^0$ is feasible in $\text{FG}(G_1)$.

Suppose for the sake of contradiction that $\|\Delta^0_{s\bullet}\|_1 > 0$. Then we can use convexity of Φ to prove the following inequalities:

$$\langle \Delta^0, \vec{D}_{2n} - f_1 \rangle \leq 0. \quad (2)$$

$$\langle \Delta^0, \vec{D}_{2n} - (f_2 - \Delta^0) \rangle > 0. \quad (3)$$

To prove (2), consider the polytope of feasible flows in $\text{FG}(G_1)$. Both f_1 and $f_1 + \Delta^0$ are in the polytope. Moreover, f_1 is the unique Φ -optimal flow in $\text{FG}(G_1)$. Since Φ is minimized at \vec{D}_{2n} , a tiny step from f_1 in the direction of $f_1 + \Delta^0$ takes us further from \vec{D}_{2n} . In other words, the angle between the vectors (f_1, \vec{D}_{2n}) and $(f_1, f_1 + \Delta^0)$ is at least 90° , implying (2).

To prove (3), consider the polytope of feasible flows in $\text{FG}(G_2)$. Both f_2 and $f_2 - \Delta^0$ are in that polytope. Moreover, f_2 is the unique Φ -optimal flow in $\text{FG}(G_2)$. Since Φ is minimized at \vec{D}_{2n} , a tiny step from $f_2 - \Delta^0$ in the direction of f_2 takes us closer to \vec{D}_{2n} . In other words, the angle between the vectors $(f_2 - \Delta^0, f_2)$ and $(f_2 - \Delta^0, \vec{D}_{2n})$ is less than 90° , implying (3).

Subtracting (3) from (2) and using the fact that $\Delta = f_2 - f_1 = \Delta^s + \Delta^t + \Delta^0$, we get

$$\begin{aligned} \langle \Delta^0, \vec{D}_{2n} - (f_2 - \Delta^0) \rangle - \langle \Delta^0, \vec{D}_{2n} - f_1 \rangle &> 0; \\ \langle \Delta^0, -(f_2 - f_1 - \Delta^0) \rangle &> 0; \\ \langle \Delta^0, \Delta^s + \Delta^t \rangle &< 0. \end{aligned} \quad (4)$$

But Δ^0 and $\Delta^s + \Delta^t$ are both subflows of Δ , so they cannot have opposite signs, on any edge, contradicting (4). Therefore, $\|\Delta^0_{s\bullet}\|_1 = 0$. \square

We now complete the proof of Theorem 4.3 (Item 2). Recall that $\Delta = \Delta^s + \Delta^t + \Delta^0$ and that Δ^s, Δ^t , and Δ^0 are subflows of Δ . From Lemmas 4.5–4.7, we get $\|\hat{f}_D(G_1) - \hat{f}_D(G_2)\|_1 = \|\Delta_{s\bullet}\|_1 = \|\Delta^s_{s\bullet}\|_1 + \|\Delta^t_{s\bullet}\|_1 + \|\Delta^0_{s\bullet}\|_1 \leq 3D$, as desired. \square

4.2 From the Degree List to the Degree Distribution

Let p_G denote the degree distribution of the graph G , i.e., $p_G(k) = |\{v : \deg_v(G) = k\}|/|V|$. Similarly, P_G denotes the *cumulative* degree distribution (CDF), i.e., $P_G(k) = |\{v : \deg_v(G) \geq k\}|/|V|$.

We can modify the extension of the degree list to get extensions of the *degree histogram* $n \cdot p_G$ or the *cumulative degree histogram* (CDH) $n \cdot P_G$. If we consider two *integral* degree lists that are at ℓ_1 distance t , then the ℓ_1 distance between their CDH's is at most t (similarly for degree histograms). However, since our extension of the degree list may produce fractional lists, we need

to extend the CDH to fractional degree lists so that the map from lists to CDHs remains Lipschitz in the ℓ_1 norm.

We do this first for the CDH; the extension of the degree histogram is an easy modification. Given an integer $k \in [D]$, let

$$[x]_k = \max\{0, \min\{1, x - (k - 1)\}\} = \begin{cases} 0 & \text{if } x \leq k - 1, \\ x - (k - 1) & \text{if } k - 1 \leq x \leq k, \\ 1 & \text{if } x \geq k. \end{cases}$$

Define the map H as follows: for a nonnegative real number a , $H(a) = ([a]_1, [a]_2, \dots, [a]_{\lceil a \rceil})$. (This is a vector of length $\lceil a \rceil$ whose ℓ_1 norm is exactly a .) Given a finite sequence $(a_1, \dots, a_n) \in [0, D]^*$, let $H(a_1, \dots, a_n) = \sum_i H(a_i)$, where we pad shorter sequence with 0's to allow summation. If the input numbers are in $[0, D]$, the sequence has length at most D .

Lemma 4.8. *The function H is 1-Lipschitz in the ℓ_1 norm. That is, $\|H(a) - H(a')\|_1 \leq \|a - a'\|_1$ for all vectors $a, a' \in [0, D]^*$. Moreover, for every graph G , $H(\text{deg-list}(G)) = n \cdot P_G$ where $n = |V_G|$.*

Proof. This follows from the fact that $H(a)$ has ℓ_1 norm a for every nonnegative real number, and equals the sequence 1^a when a is an integer. \square

Given H , which extends transforms degree lists to the CDH, we can obtain an extension of the degree histogram via $\text{hist}_D(a) = H_D(a)$, and $\text{hist}_i(a) = H_i(a) - H_{i+1}(a)$ for $i < D$. This increases ℓ_1 distances by at most an additional factor of 2.

Theorem 4.9. *The map*

$$G \mapsto \text{hist}(\hat{f}_D(G))$$

extends the degree histogram (as a map from $(\mathcal{G}^D, d_{\text{node}})$ to ℓ_1^D) to \mathcal{G} , with stretch at most 3.

4.3 Differentially Private Approximations to the Degree Distribution

There are two natural approaches to using the extension of deg-list to release an approximate degree distribution. First, we may add noise D/ϵ to each entry of the sorted degree list, and project (and/or) remove noise as in [12, 17, 24]. The second is to release the D -bounded degree histogram and add noise. The error of the first approach is difficult to bound analytically, and so we adopt the second here.

Given a degree threshold D , consider the following mechanism:

Algorithm 1: Noisy Degree Histogram(G, ϵ, D)

- 1 $Y_i \sim \text{Lap}(6D/\epsilon)$ for $i = 1, \dots, D$;
 - 2 **return** $\mathcal{A}_D(G) = \text{hist}(\hat{f}_D(G)) + (Y_1, \dots, Y_D)$;
-

(We only need to release the first D entries of hist , since the remaining entries are always 0.)

This mechanism introduces two sources of error: the extension error $\hat{f}(G) - \text{deg-list}(G)$ and the random noise $\vec{Y} = (Y_1, \dots, Y_d)$. The noise component is easy to understand and bound. How can we characterize the error introduced by the extension?

Lemma 4.10. *For any graph G and threshold D , the extension's ℓ_1 error satisfies*

$$n \sum_{i>D} P_G(i) \leq \|\hat{f}_D(G) - \text{deg-list}(G)\|_1 \leq 2n \sum_{i>D} P_G(i).$$

Proof. Recall that $\hat{f}(G)$ has minimal ℓ_1 error $\|\hat{f}_D(G) - \text{deg-list}(G)\|_1$ among all D -bounded real vectors that are consistent with a weighted graph. In particular, one can consider a graph G' which is obtained by removing $\text{deg}_v - D$ edges for each vertex v with degree greater than D . Each edge removal causes a change of 2 in $\text{deg-list}(G)$ in the ℓ_1 norm. The number of edges removed is $\sum_{v: \text{deg}_v > D} (\text{deg}_v - D)$. An alternative formula for this sum can be obtained by summing over degrees instead of vertices:

$$\sum_{v: \text{deg}_v > D} (\text{deg}_v - D) = \sum_{i>D} nP_G(i)$$

(since each vertex v contributes $\max(0, \text{deg}_v - D)$ to the sum). Multiplying by 2 yields the desired upper bound.

To prove the lower bound, note that the vector $\hat{f}_D(G)$ is always less, coordinatewise, than the simple projection that replaces the degree deg_v of each vertex v by $\min(D, \text{deg}_v)$. The ℓ_1 error of $\hat{f}(G)$ (or indeed of any function that projects onto a set of vectors with entries bounded by D) is therefore at least $\sum_{v: \text{deg}_v > D} (\text{deg}_v - D) = n \sum_{i>D} P_G(i)$. \square

Combining the two previous lemmas with the fact that the expected absolute value of each Y_i is $4D/\epsilon$, we obtain the following theorem.

Theorem 4.11. *The expected ℓ_1 error of algorithm \mathcal{A}_D on input G is at most $2n \sum_{i>D} P_G(i) + \frac{6D^2}{\epsilon}$.*

This theorem bounds the error of the algorithm for a given degree threshold D . In the sequel, we show how we can select a (nearly) optimal threshold differentially privately.

5 Exponential Mechanism For Scores With Varying Sensitivity

The exponential mechanism of McSherry and Talwar [31] is a basic tool for designing differentially private algorithms. We present here a generalization for score functions with different sensitivities.

Suppose the data set comes from a universe U equipped with an neighbor relation (e.g., Hamming or set-difference distance for standard data sets, or vertex distance on graphs). We assume that the set of possible answers is finite and index it by elements of $[k]$. Given a collection of functions q_1, \dots, q_k from U to \mathbb{R} and a private data set $x \in U$, the goal is to minimize $q_i(x)$, that is, to find an index \hat{i} such that $q_{\hat{i}}(x) \approx \min_i q_i(x)$. Define

$$\Delta_i \stackrel{\text{def}}{=} \max_{x, x' \in U \text{ adjacent}} |q_i(x) - q_i(x')| \quad \text{and} \quad \Delta_{\max} \stackrel{\text{def}}{=} \max_i \Delta_i.$$

The exponential mechanism achieves the following accuracy guarantee: for every $\beta > 0$, with probability $1 - \beta$, the output \hat{i} satisfies $q_{\hat{i}}(x) \leq \min_i q_i(x) + \Delta_{\max} \cdot \frac{2 \log(k/\beta)}{\epsilon}$.

A limitation of this guarantee is that it depends on the maximum sensitivity of the score functions $q_i(\cdot)$. In the context of threshold selection for graph algorithms, such a guarantee is meaningless for sparse graphs. This poor utility bound is not merely an artifact of the analysis.

Algorithm 2: Generalized Exponential Mechanism

Input: Data set x from universe U , parameters $\beta \in (0, 1)$ and $\epsilon > 0$,
score functions q_1, \dots, q_k from U to \mathbb{R} .

- 1 Set $t = \frac{2 \log(k/\beta)}{\epsilon}$;
- 2 **for** $i = 1$ **to** k **do**
- 3 $\Delta_i \stackrel{\text{def}}{=} \max_{x, x' \in U \text{ adjacent}} |q_i(x) - q_i(x')|$. /* An upper bound on Δ_i suffices. */
 /* Generally, Δ_i is known exactly. */
- 4 **for** $i = 1$ **to** k **do**
- 5 $s(i) \leftarrow \max_{j \in [k]} \frac{(q_i(x) + t\Delta_i) - (q_j(x) + t\Delta_j)}{\Delta_i + \Delta_j}$ /* $s(i)$ has sensitivity at most 1. */
- 6 **return** $\hat{i} \leftarrow \text{ExponentialMechanism}(s(i), i \in [k], \epsilon)$;

The problem is inherent in the algorithm. For example, consider the setting with $k = 2$, where the two score functions have sensitivity $\Delta_1 = 1$ and $\Delta_2 \gg 1$. Further, consider a data set x with $q_1(x) = 0$ and $q_2(x) = \Delta_2/\epsilon$. On input x , the exponential mechanism will select $\hat{i} = 2$ with constant probability, resulting in an excess error of Δ_2/ϵ , which may be arbitrarily larger than Δ_1 .

In contrast, we give an algorithm whose excess error scales with the sensitivity of the *optimal* score function Δ_{i^*} , where $i^* = \operatorname{argmin}_i q_i(x)$. Our mechanism requires as input an upper bound β on the desired probability of a bad outcome; the algorithm's error guarantee depends on this β .

Theorem 1.4 (Formal). *For all parameters $\beta \in (0, 1)$, $\epsilon > 0$, the generalized exponential mechanism (Algorithm 2) is $(\epsilon, 0)$ -differentially private (with respect to the neighbor relation on U). For all inputs x , the output \hat{i} satisfies*

$$q_{\hat{i}}(x) \leq \min_i \left(q_i(x) + \Delta_i \cdot \frac{4 \log(k/\beta)}{\epsilon} \right). \quad (5)$$

In particular, our algorithm is competitive with the sensitivity of the true minimizer $i^* = \operatorname{argmin}_i q_i(x)$ (since the right-hand side of (5) is at most $q_{i^*}(x) + \Delta_{i^*} \cdot \frac{4 \log(k/\beta)}{\epsilon}$). In the case that all the Δ_i 's are the same, our algorithm simplifies to running the usual exponential mechanism with $\epsilon' = \epsilon/2$; this justifies the “generalized” name.

The intuition behind the algorithm is as follows: since the score function q has different sensitivity for each i , we would like to find an alternative score function which is less sensitive. One simple score would be to compute, for each j , the *distance*, in the space of data sets, from the input x to the nearest data set y in which $q_j(y)$ is smallest among the values $\{q_i(y)\}_{i \in [k]}$ (this idea is inspired by the GWAS algorithms of [14]). This score has two major drawbacks: first, it is hard to compute in general; second, more subtly, it will tend to favor indices j with very high sensitivity (since they become optimal with relatively few changes to the data).

Instead, we use a substitute measure which is both easy to compute (given the scores $q_i(x)$ for $i \in [k]$) and appropriately penalizes scores with large sensitivity. Given a value $t > 0$ (to be set later), define the *normalized score* as

$$s(i; x) = \max_{j \in [k]} \frac{(q_i(x) + t\Delta_i) - (q_j(x) + t\Delta_j)}{\Delta_i + \Delta_j} = \max_{j \in [k]} \left(\frac{q_i(x) - q_j(x)}{\Delta_i + \Delta_j} + t \cdot \frac{\Delta_i - \Delta_j}{\Delta_i + \Delta_j} \right). \quad (6)$$

The first term inside the maximum on the right-hand side is an approximation to the Hamming distance from x to the nearest data set y where score $q_j(\cdot)$ becomes smaller than $q_i(\cdot)$. The second term (containing t and independent of the data set) penalizes indices i with larger sensitivity.

We obtain an index \hat{i} by running the usual exponential mechanism on the normalized scores $s(i)$. Our first lemma bounds the sensitivity of the normalized score.

Lemma 5.1. *For each i , and for any $t \in \mathbb{R}$, the normalized score $s(i; \cdot)$ has sensitivity at most 1.*

Proof. First, fix indices $i, j \in [k]$. The ratio $\frac{q_i(x) - q_j(x)}{\Delta_i + \Delta_j}$ has sensitivity at most 1 since $q_i(\cdot)$ and $q_j(\cdot)$ can vary by at most Δ_i and Δ_j , respectively, when x changes to an adjacent data set. As long as t does not depend on x , the function $s(i; \cdot)$ is a maximum of sensitivity-1 functions, which means its sensitivity is at most 1. \square

Proof of Theorem 1.4. The algorithm sets $t = 2 \ln(k/\beta)/\epsilon$, regardless of the data x . Since the normalized scores have sensitivity at most 1, the application of the usual exponential mechanism (or its more efficient alternative, “report noisy min”) is $(\epsilon, 0)$ -differentially private.

To analyze utility, let \tilde{i} denote the index that minimizes the penalized score $q_i(x) + t\Delta_i$. Then

$$s(\tilde{i}; x) = 0,$$

since each of the terms in the maximum defining s is nonpositive for \tilde{i} (and the term for $j = \tilde{i}$ is 0). By the usual analysis of the exponential mechanism, we have that with probability at least $1 - \beta$,

$$s(\hat{i}; x) \leq \underbrace{s(\tilde{i}; x)}_0 + \frac{2 \ln(k/\beta)}{\epsilon}.$$

Now consider an arbitrary index j . Since $s(\hat{i}; x)$ is at least $\frac{(q_i(x) + t\Delta_i) - (q_j(x) + t\Delta_j)}{\Delta_i + \Delta_j}$, we can multiply by $\Delta_i + \Delta_j$ to obtain:

$$\begin{aligned} q_i(x) &\leq q_j(x) + t(\Delta_j - \Delta_i) + \frac{2 \ln(k/\beta)}{\epsilon} \cdot (\Delta_i + \Delta_j) \\ &= q_j(x) + \Delta_j \left(\frac{2 \ln(k/\beta)}{\epsilon} + t \right) + \Delta_i \left(\frac{2 \ln(k/\beta)}{\epsilon} - t \right). \end{aligned}$$

Substituting $t = \frac{2 \ln(k/\beta)}{\epsilon}$ yields the desired result. \square

5.1 Threshold Selection for Lipschitz Extensions

Suppose we have a collection of candidate functions $\{f_{\Delta_i}\}_{i \in [k]}$ for approximating a function f , each with sensitivity Δ_i . Moreover, the approximation functions are all underestimates, that is,

$$f_{\Delta_i}(G) \leq f(G) \quad \text{for all } G \text{ and } \Delta_i.$$

Let

$$err(\Delta_i) = |f(G) - f_{\Delta_i}(G)| + \Delta_i/\epsilon.$$

This is a simple proxy for the (expected) error in approximating $f(G)$ that one gets by using $f_{\Delta_i}(G) + \text{Lap}(\Delta_i/\epsilon)$. It exaggerates the expected error by a factor of at most 2, since the expected error is at most $err(\Delta_i)$ by the triangle inequality, and at least $\min(|f(G) - f_{\Delta_i}(G)|, \Delta_i/\epsilon)$.

The functions $err(\Delta_i)$ don't necessarily have bounded sensitivity (since we make no assumption on how f varies). However, the differences $err(\Delta_i) - err(\Delta_j)$ do have sensitivity at most $\Delta_i + \Delta_j$, which allows us to employ the generalized exponential mechanism (alternatively, since the functions f_{Δ_i} are all underestimates, we may use $q_i(x) = -f_{\Delta_i}(G) + \Delta/\epsilon$).

Corollary 5.2. *Running the generalized exponential mechanism with score $q_i(x) = err(\Delta_i)$ and sensitivities Δ_i is differentially private and yields a threshold $\hat{\Delta}$ such that, for every $\Delta^* \in \{\Delta_i\}$, with probability at least $1 - \beta$,*

$$err(\hat{\Delta}) \leq err(\Delta^*) + \frac{4\Delta^* \log(k/\beta)}{\epsilon} \leq err(\Delta^*) \cdot O\left(\ln\left(\frac{k}{\beta}\right)\right).$$

Selecting from a continuous interval of thresholds The extensions we consider satisfy a further guarantee of *monotonicity*, namely, if $\Delta_1 < \Delta_2$, we have

$$f_{\Delta_1}(G) \leq f_{\Delta_2}(G) \leq f(G). \quad (7)$$

If we want to select among an interval $[1, \Delta_{max}]$ of possible thresholds, then this guarantee ensures that selecting among the powers of a fixed constant (e.g., $1, 2, 4, \dots, 2^{\lceil \log_2(\Delta_{max}) \rceil}$) will still give a multiplicative approximation to be best choice of Δ , since for all values of $\Delta \geq 0$,

$$err(\Delta) \leq 2err(\Delta/2).$$

We obtain the following proposition.

Proposition 5.3. *If the collection of functions $\{f_{\Delta}\}_{\Delta \in [1, \Delta_{max}]}$ forms a monotone family of approximations to f (as in (7)), then applying the generalized exponential mechanism to the powers of 2 in the interval $[1, \Delta_{max}]$ yields a threshold $\hat{\Delta}$ such that, for every $\Delta^* \in [1, \Delta_{max}]$, with probability at least $1 - \beta$,*

$$err(\hat{\Delta}) = err(\Delta^*) \cdot O\left(\ln \ln(\Delta_{max}) + \ln \frac{1}{\beta}\right).$$

This generalizes and improves on the result of Chen and Zhou [6], who gave a method for selecting a sensitivity threshold that was specific to their Lipschitz extensions and within a $\log(n)$ multiplicative factor (as opposed to $\log \log n$) of the optimal error.

5.2 Selecting a Threshold for the Degree Distribution

Consider the algorithm for releasing the degree distribution discussed in Section 4. Recall that the algorithm's ℓ_1 error is at most

$$err(D) \stackrel{\text{def}}{=} \|\hat{f}_D(G) - \text{deg-list}(G)\|_1 + 6D^2/\epsilon. \quad (8)$$

This error function is closely related to the error of the approximation to the number of edges from Kasiviswanathan et al. [20]. Specifically, let $g(G)$ denote the number of edges in G , and g_D denote the Lipschitz extension of g from \mathcal{G}^D to \mathcal{G} . Then

$$g_D(G) = \|\hat{f}_D(G)\|_1 \quad \text{and} \quad g(G) - g_D(G) = \|\hat{f}_D(G) - \text{deg-list}(G)\|_1.$$

We can therefore use the process above for selecting a threshold for a one-dimensional function to select a threshold for releasing the degree distribution.

Proposition 5.4. *Given a graph G and parameter ϵ , let $D^* = \operatorname{argmin}_{D \in [1, n]} \operatorname{err}(D)$. Applying the generalized exponential mechanism with $q_i(G) = \operatorname{err}(2^i)$, for $i \in \{1, 2, 4, \dots, 2^{\lceil \log_2(n) \rceil}\}$ is $(\epsilon, 0)$ -differentially private and yields a threshold \hat{D} such that, for every $\Delta^* \in [1, \Delta_{max}]$, with probability at least $1 - \beta$,*

$$E\left(\operatorname{err}(\hat{D})\right) \leq 2\operatorname{err}(D^*) + \frac{8D^* \ln(\ln(n)/\beta)}{\epsilon} = \operatorname{err}(D^*) \cdot O\left(\ln \ln(n) + \ln \frac{1}{\beta}\right).$$

6 Error Analysis on α -Decaying Graphs

Our techniques provide a significantly more accurate way to release the degree distributions of graphs while satisfying node-level differential privacy. To illustrate this, we study the accuracy of our method on graphs that satisfy α -decay, a mild condition on the tail of the degree distribution.

6.1 α -Decay

Recall that $\bar{d}(G) = 2|E|/|V|$ is the average degree of G .

Assumption 6.1 (α -decay). *Fix $\alpha \geq 1$. A graph G satisfies α -decay if for all¹ real numbers $t > 1$, $P_G(t \cdot \bar{d}) \leq t^{-\alpha}$.*

Note that *all* graphs satisfy 1-decay (by Markov’s inequality). The assumption is nontrivial for $\alpha > 1$, but it is nevertheless satisfied by almost all widely studied classes of graphs. So-called “scale-free” networks (those that exhibit a heavy-tailed degree distribution) typically satisfy α -decay for $\alpha \in (1, 2)$. Random graphs satisfy α -decay for essentially arbitrarily large α since their degree distributions have tails that decay exponentially (more precisely, for any α we can find a constant c_α such that, with high probability, α -decay holds when $t > c_\alpha$). Regular graphs satisfy the assumption with $\alpha = \infty$. The following lemma bounds the number of *edges* adjacent to nodes with degree above a given threshold.

Lemma 6.2. *Consider a graph G on n nodes that satisfies α -decay for $\alpha > 1$, and let $D > \bar{d}(G)$. Then*

$$\sum_{v: \deg_v(G) > D} \leq \frac{\bar{d}(G)^\alpha}{(\alpha + 1)D^{\alpha-1}} \cdot n.$$

6.2 Error Analysis

Kasiviswanathan et al. [20] gave algorithms for releasing the degree distribution using a projection-based technique. Their algorithm required knowledge of the decay parameter α (which was used to select the projection threshold). They bounded the ℓ_1 error of their algorithm in estimating the degree distribution, and showed that it went to 0 as long as $\alpha > 2$ and \bar{d} was polylogarithmic in n . More precisely, they gave an expected error bound of

$$\mathbb{E} \|\hat{p} - p_G\|_1 = \tilde{O}\left(\bar{d}^{\frac{3\alpha}{\alpha+1}} / \left(\epsilon^2 n^{\frac{\alpha-2}{\alpha+1}}\right)\right).$$

¹Our results hold even when this condition is satisfied only for sufficiently large t . For simplicity, we use a stronger assumption in our presentation.

Combining the noisy Lipschitz extension of the degree histogram (Theorem 4.11) with the threshold selection algorithm (Proposition 5.4), we get an algorithm $\mathcal{A}_{\text{combo}}$ with much better accuracy guarantees that, additionally, does not need to know the parameter α .

Algorithm 3: Degree Histogram Estimation For Unknown Threshold

- 1 $\hat{D} \leftarrow$ Generalized Exponential Mechanism($q_D(\cdot), D \in \{1, 2, 4, \dots, 2^{\lfloor \log n \rfloor}\}$) using score $q_D(G) = \text{err}(D)$ and sensitivity bound D ;
 - 2 $\hat{h} \leftarrow \hat{f}_{\hat{D}}(G) + (Y_1, \dots, Y_{\hat{D}})$ where $Y_i \sim \text{Lap}(4\hat{D}/\epsilon)$ i.i.d. ;
 - 3 **return** $\hat{p} = \hat{h}/\|\hat{h}\|_1$;
-

Theorem 6.3. *Given inputs $G \in \mathcal{G}$ and $\epsilon > 0$, the algorithm $\mathcal{A}_{\text{combo}}$ produces an estimate \hat{p} such that, if G satisfies α decay for $\alpha > 1$, then*

$$\mathbb{E} \|\hat{p} - p_G\|_1 = O\left(\bar{d}^{\frac{2\alpha}{\alpha+1}} (\ln \ln n) / (\epsilon n)^{\frac{\alpha-1}{\alpha+1}}\right).$$

In particular, this error is $o(1)$ as n goes to infinity if $\alpha > 1$ and $\bar{d}^{\frac{2\alpha}{\alpha+1}} = o(\epsilon n)$.

Proof. Fix a graph G that satisfies α decay for $\alpha > 1$, and let \bar{d} denote its average degree. Conditioned on selecting a given degree threshold D , Theorem 4.11 guarantees that the ℓ_1 error of our algorithm in estimating $\text{hist}(\text{deg-list}(G)) = n \cdot p_G$ is at most $\text{err}(D) = 2n \sum_{i>D} P_G(i) + \frac{6D^2}{\epsilon}$ (defined as in (8)).

Although the true size of the graph n is not known to the algorithm it is convenient to divide everything by n so that we can compare to the true degree distribution p_G . Let $\tilde{p} = \hat{h}/n$ denote the estimate of p_G one gets by normalizing the estimated degree histogram by the true vertex count n rather than $\|\hat{h}\|_1$. We will account for the estimation of n at the end of the proof. Dividing by n , we get a bound of the form $\frac{\text{err}(D)}{n} = O\left(\sum_{i>D} P_G(i) + \frac{D^2}{\epsilon n}\right)$ on the error in estimating p_G . By Lemma 6.2, this bound is at most $O\left(\bar{d}^\alpha / ((\alpha+1)D^{\alpha-1}) + D^2/(n\epsilon)\right)$. In particular, if we set $D^* = (\bar{d}^\alpha \epsilon n)^{1/(\alpha+1)}$ (which makes the two terms in the sum equal) then the expected ℓ_1 error of \tilde{p} is at most

$$\frac{\text{err}(D^*)}{n} = O(\bar{d}^{\frac{2\alpha}{\alpha+1}} / (\epsilon n)^{\frac{\alpha-1}{\alpha+1}}).$$

In the algorithm, we do not select D^* but rather a differentially private alternative \hat{D} . By the law of conditional expectations, the overall expected error of \tilde{p} is at most the expectation of $\frac{\text{err}(\hat{D})}{n}$, that is,

$$\mathbb{E} \|\tilde{p} - p_G\|_1 = \mathbb{E}_{\hat{D}} \left(\mathbb{E}_{\text{noise}} \left(\|\tilde{p} - p_G\|_1 \mid \hat{D} \right) \right) \leq \frac{1}{n} \mathbb{E}_{\hat{D}} \left(\text{err}(\hat{D}) \right).$$

By Proposition 5.4, the expectation of $\text{err}(\hat{D})$ is at most $2\text{err}(D^*) + 8D^* \ln \ln(n)/\epsilon$. For $\alpha < \infty$, the reference threshold D^* is polynomially large in n . Thus, the first term $\text{err}(D^*)$ (which is at least $(D^*)^2/\epsilon$) dominates the second term, and the final error bound is

$$\mathbb{E} \|\tilde{p} - p_G\|_1 = O(\bar{d}^{\frac{2\alpha}{\alpha+1}} / (\epsilon n)^{\frac{\alpha-1}{\alpha+1}}).$$

Finally, we analyze the difference between $\mathcal{A}_{\text{combo}}$ and \mathcal{A}' . Let \hat{n} denote the estimated number of edges in G , that is $\hat{n} = \|\hat{h}\|_1$. Note that for any given realization of the algorithm's random

choices, if \tilde{p} is a good approximation to the true distribution p_G , then \hat{n} must be good estimate of the true number of vertices:

$$|\hat{n} - n| = | \|\hat{h}\|_1 - n | \leq n | \|\tilde{p}\|_1 - 1 | \leq n \|\tilde{p} - p_G\|_1.$$

This allows us to bound the difference between \hat{p} and \tilde{p} . Since $\tilde{p} - \hat{p} = \hat{p} \left(\frac{\hat{n}}{n} - 1 \right)$, the ℓ_1 norm of $\tilde{p} - \hat{p}$ is at most $\|\hat{p}\|_1 \|\tilde{p} - p_G\|_1 = \|\tilde{p} - p_G\|_1$. Thus, the error of \hat{p} in estimating p_G is never more than twice the error of \tilde{p} :

$$\|\hat{p} - p_G\|_1 \leq 2\|\tilde{p} - p_G\|_1 \quad \text{and thus} \quad \mathbb{E} \|\hat{p} - p_G\|_1 = O(\bar{d}^{\frac{2\alpha}{\alpha+1}} / (\epsilon n)^{\frac{\alpha-1}{\alpha+1}}). \quad \square$$

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A Doubling dimension of graph metrics

Definition A.1. *The doubling dimension ρ of a metric space (X, d) is the smallest integer such that every ball in X can be covered using at most 2^ρ balls of half the radius.*

Consider the set \mathcal{G}_n of graphs on at most n vertices. If we equip \mathcal{G}_n with the edge adjacency metric, we get a set essentially equivalent to the $\binom{n}{2}$ -dimensional Hamming cube (in fact, a union of n different Hamming cubes corresponding to graphs of sizes $1, 2, \dots, n$). This metric has doubling dimension $\Theta(n^2)$.

Intuitively, the doubling dimension of the vertex-adjacency metric on \mathcal{G}_n^D should be similar. We sketch a weaker statement here, namely that the doubling dimension is $\Omega(n)$. This bound shows that constructions with stretch bounded by the doubling dimension still have very high stretch when used on the vertex metric.

Lemma A.2. *The doubling dimension of the vertex-adjacency metric on \mathcal{G}_n^D for $D \geq 1$ is $\Omega(n)$. If we collapse the set \mathcal{G}_n^D by identifying isomorphic graphs, then the statement continues to hold for $D \geq 4$.*

Proof Sketch. Assume n is even, w.l.o.g. To prove the theorem, we embed the Hamming cube $Ham_{n/2}$ into \mathcal{G}_n^D , which shows that \mathcal{G}_n^D has doubling dimension $\Omega(n)$. Let G_0 be a uniformly random regular graph of degree 3 on $n/2$ vertices. For every subset $S \subseteq [n/2]$, let G_S be the graph on $n/2 + |S|$ vertices obtained by starting from G_0 and adding one vertex for each element in S and connecting it to the corresponding vertex in G_0 .

The vertex distance between two such graph G_S and G_T is $\Omega(|S\Delta T|)$, as with the Hamming metric, as long as S and T are sufficiently far from each other. This is sufficient to prove the main result as we may select S and T from an error-correcting code with linear rate and minimum distance.

A complete proof is delicate, since one must account for the possibility that one can get from G_S to a graph that is isomorphic to a subgraph of G_T by deleting fewer than $|S\Delta T|$ vertices from G_0 . We omit the details. \square